

Second-order linear differential equations with two irregular singular points of rank three: the characteristic exponent

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Abstract

For a second-order linear differential equation with two irregular singular points of rank three, multiple Laplace-type contour integral solutions are considered. An explicit formula in terms of the Stokes multipliers is derived for the characteristic exponent of the multiplicative solutions. The Stokes multipliers are represented by converging series with terms for which limit formulas as well as more detailed asymptotic expansions are available. Here certain new, recursively known coefficients enter, which are closely related to but different from the coefficients of the formal solutions at one of the irregular singular points of the differential equation. The coefficients of the formal solutions then appear as finite sums over subsets of the new coefficients. As a by-product, the leading exponential terms of the asymptotic behaviour of the late coefficients of the formal solutions are given, and this is a concrete example of the structural results obtained by Immink in a more general setting. The formulas displayed in this paper are not of merely theoretical interest, but they also are complete in the sense that they could be (and have been) implemented for computing accurate numerical values of the characteristic exponent, although the computational load is not small and increases with the rank of the singular point under consideration.

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1. Introduction

Let us consider the differential equation

$$z^2 f'' + z f' - \left[\sum_{m=1}^6 D_m z^{-m} + L^2 + \sum_{m=1}^6 B_m z^m \right] f(z) = 0 \quad (1.1)$$

with the thirteen parameters $D_1, \dots, D_6, L, B_1, \dots, B_6$, which for simplicity of presentation are assumed to be real. This differential equation has two irregular singular points, each of rank 3 (if $D_6 \neq 0$ and $B_6 \neq 0$, respectively), at the origin and at infinity. Without loss of generality, one of the parameters except L could be set equal to 1. We assume that B_6 is positive, again for simplicity of presentation, and that L is not negative.

At infinity, there are formal power series solutions

$$f_{\infty 1}^{\text{asy}}(z) = \exp(P(z))z^{-\tau(1)} \sum_{n=0}^{\infty} a_n(1)z^{-n}, \quad (1.2)$$

$$f_{\infty 2}^{\text{asy}}(z) = \exp(-P(z))z^{-\tau(-1)} \sum_{n=0}^{\infty} a_n(-1)z^{-n}, \quad (1.3)$$

where the various quantities are determined by

$$P(z) = p_3 z^3 + p_2 z^2 + p_1 z, \quad (1.4)$$

$$p_3 = \frac{1}{3}\sqrt{B_6}, \quad p_2 = \frac{1}{12}B_5/p_3, \quad p_1 = \frac{1}{6}(B_4 - 4p_2^2)/p_3, \quad (1.5)$$

$$\tau(\kappa) = \frac{3}{2} - \frac{1}{6} \frac{B_3}{\kappa p_3} + \frac{2}{3} \frac{p_1 p_2}{\kappa p_3}, \quad (1.6)$$

$$\kappa \in \{1, -1\} \quad (1.7)$$

and where the coefficients

$$a_n(\kappa) = a_n \quad (1.8)$$

are known recursively by

$$a_0 = 1 \quad (1.9)$$

and

$$\begin{aligned} 6\kappa p_3 n a_n &= [-4\kappa p_2(\tau(\kappa) + n - 2) + p_1^2 - B_2]a_{n-1} \\ &+ [-2\kappa p_1(\tau(\kappa) + n - \frac{5}{2}) - B_1]a_{n-2} + (\tau(\kappa) + n - 3 - L)(\tau(\kappa) + n - 3 + L)a_{n-3} \\ &- \sum_{m=1}^6 D_m a_{n-m-3}, \end{aligned} \quad (1.10)$$

$$n = 1, 2, \dots, \quad (a_{-8} = a_{-7} = \dots = a_{-1} = 0).$$

The formal solutions are asymptotic expansions as $z \rightarrow \infty$ in appropriate sectors of the complex plane.

At the origin, there are analogical formal solutions, which may be obtained from those at infinity by the simultaneous replacements $z \leftrightarrow 1/z$, $B_m \leftrightarrow D_m$, $m = 1, \dots, 6$. This symmetry is our main reason for choosing just (1.1) as the standard form for a differential equation with two irregular singular points.

In the ring-shaped region $0 < |z| < \infty$ we have (convergent) Floquet solutions $f^{(\omega)}(z)$ and $f^{(-\omega)}(z)$, where

$$f^{(\omega)}(z) = z^\omega \sum_{n=-\infty}^{+\infty} c_n^{(\omega)} z^n, \quad (1.11)$$

which are linearly independent if 2ω is not equal to an integer. Here the coefficients $c_n^{(\omega)}$ obey the recurrence relation

$$-\sum_{m=1}^6 B_m c_{n-m}^{(\omega)} + (-L + \omega + n)(L + \omega + n) c_n^{(\omega)} - \sum_{m=1}^6 D_m c_{n+m}^{(\omega)} = 0, \quad (1.12)$$

and we want to normalize them by choosing

$$c_0^{(\omega)} = 1. \quad (1.13)$$

The requirement that the power series converge determines, modulo 1, the possible values of the characteristic exponent (or circuit exponent or Floquet exponent) ω . The problem to compute the characteristic exponent appears also and is best known in the context of Hill's differential equation [22]. There are methods to compute the characteristic exponent numerically, which require, for $\omega = 0$, the evaluation of the infinite determinant [20] associated with (1.12), or numerical integration of the differential equation along a suitable contour [21], or numerical solution of an eigen-value problem [14].

This paper develops an entirely different method for evaluating the characteristic exponent. We obtain an explicit formula in terms of quantities which are essentially the Stokes multipliers, and these are given explicitly as convergent series, the terms of which are represented by asymptotic expansions. Here certain recursively known coefficients enter which are closely related to but different from the coefficients of the formal solutions (1.2)-(1.3).

Although some of the other authors concerned with irregular singular points of rank larger than one [1, 2, 4-6, 8-10, 12, 13, 15, 18, 19] consider the general case of arbitrary rank, we here prefer to restrict our attention to rank three. This is already general enough to give an impression of what can be expected in the case of even higher rank. On the other hand, it is still simple enough so that we can, for the relevant quantities, obtain explicit expressions which are not only theoretically interesting but can also be implemented (and have been implemented) for numerical evaluation. This work may be viewed as an attempt to extend [3], which was useful for rank one, to the much more complicated case of higher rank.

2. Laplace contour integral solutions

We try to apply the classical method of multiple Laplace contour integral solutions [7] and write for a solution of (1.1)

$$f(z) = z^\lambda (2\pi i)^{-3} \int_{C_{t_2}} \int_{C_{t_1}} \int_{C_s} \exp(z^3 s + z^2 t_1 + z t_2) v(s, t_1, t_2) ds dt_1 dt_2, \quad (2.1)$$

where a power factor with a still arbitrary parameter λ has been included in view of later benefits. To derive the appropriate weight function $v(s, t_1, t_2)$ is a somewhat lengthy but not principally difficult procedure. We therefore give two lemmata stating the results of this procedure and postpone the proofs to a later section.

Lemma 1. *The weight function $v(s, t_1, t_2)$ has to be a solution of the partial differential equation*

$$\begin{aligned} & 9(s^2 - s_0^2)(\partial^4 v / \partial s^4) + 12(st_1 - s_0 t_{10})(\partial^4 v / \partial s^3 \partial t_1) + 6(st_2 - s_0 t_{20})(\partial^4 v / \partial s^3 \partial t_2) \\ & + 4(t_1^2 - t_{10}^2)(\partial^4 v / \partial s^2 \partial t_1^2) + (t_2^2 - t_{20}^2)(\partial^4 v / \partial s^2 \partial t_2^2) + 4(t_1 t_2 - t_{10} t_{20})(\partial^4 v / \partial s^2 \partial t_1 \partial t_2) \\ & + ([81 - 6\lambda]s + [B_3 - 4t_{10} t_{20}])(\partial^3 v / \partial s^3) + ([52 - 4\lambda]t_1 + [B_2 - t_{20}^2])(\partial^3 v / \partial s^2 \partial t_1) \\ & + ([25 - 2\lambda]t_2 + B_1)(\partial^3 v / \partial s^2 \partial t_2) + ([\lambda - 12]^2 - L^2)(\partial^2 v / \partial s^2) - D_1(\partial^2 v / \partial s \partial t_1) \\ & - D_2(\partial^2 v / \partial s \partial t_2) + D_3(\partial v / \partial s) + D_4(\partial v / \partial t_1) + D_5(\partial v / \partial t_2) - D_6 v = 0 \end{aligned} \quad (2.2)$$

with two finite singular points at $(s, t_1, t_2) = (\kappa s_0, \kappa t_{10}, \kappa t_{20})$ where

$$t_{20} = p_1, \quad t_{10} = p_2, \quad s_0 = p_3, \quad (2.3)$$

and the contours for each variable have to satisfy the condition that a certain lengthy expression, bilinear in K and v or their partial derivatives, have the same value at the start and the end of the contour.

Lemma 2. For $\kappa \in \{1, -1\}$, there are appropriate weight functions $v = V(\kappa; s, t_1, t_2)$ which at the singular point $(s, t_1, t_2) = (\kappa s_0, \kappa t_{10}, \kappa t_{20})$ have the power series expansion

$$\begin{aligned} & V(\kappa; s, t_1, t_2) \\ &= \sum_{m=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} A(\kappa; m, n_1, n_2) (s - \kappa s_0)^{\mu(\kappa)+m} (t_1 - \kappa t_{10})^{-\nu_1-n_1} (t_2 - \kappa t_{20})^{-\nu_2-n_2}, \end{aligned} \quad (2.4)$$

where the A -coefficients satisfy a certain recurrence relation and the exponents are related by

$$3\mu(\kappa) - 2\nu_1 - \nu_2 = \lambda + \tau(\kappa) - 6. \quad (2.5)$$

An appropriate set of coefficients is

$$A(\kappa; m, n_1, n_2) = \Gamma(-\mu(\kappa) - m) \Gamma(\nu_1 + n_1) \Gamma(\nu_2 + n_2) b(\kappa; m, n_1, n_2), \quad (2.6)$$

where the new coefficients $b(\kappa; m, n_1, n_2)$ are given by the recurrence relation

$$\begin{aligned} & 6\kappa s_0(3m - 2n_1 - n_2)b(\kappa; m, n_1, n_2) \\ &= [(3m - 2n_1 - n_2 + \tau(\kappa) - 3)^2 - L^2]b(\kappa; m - 1, n_1, n_2) \\ &+ [-4\kappa t_{10}(3m - 2n_1 - n_2 + \tau(\kappa) - 2) + t_{20}^2 - B_2]b(\kappa; m - 1, n_1 - 1, n_2) \\ &+ [-2\kappa t_{20}(3m - 2n_1 - n_2 + \tau(\kappa) - \frac{5}{2}) - B_1]b(\kappa; m - 1, n_1, n_2 - 1) \\ &- D_1b(\kappa; m - 2, n_1 - 1, n_2) - D_2b(\kappa; m - 2, n_1, n_2 - 1) - D_3b(\kappa; m - 2, n_1, n_2) \\ &- D_4b(\kappa; m - 3, n_1 - 1, n_2) - D_5b(\kappa; m - 3, n_1, n_2 - 1) - D_6b(\kappa; m - 3, n_1, n_2) \end{aligned} \quad (2.7)$$

with the initial conditions

$$b(\kappa; 0, 0, 0) = 1,$$

$$b(\kappa; m, n_1, n_2) = 0 \quad \text{if} \quad 3m - 2n_1 - n_2 = 0 \quad \text{for} \quad (m, n_1, n_2) \neq (0, 0, 0), \quad (2.8)$$

$$(b(\kappa; m, n_1, n_2) = 0 \quad \text{if} \quad m < 0 \quad \text{or} \quad n_1 < 0 \quad \text{or} \quad n_2 < 0)$$

and with

$$b(\kappa; 0, n_1, n_2) = 0 \quad \text{for} \quad (n_1, n_2) \neq (0, 0), \quad (2.9)$$

$$b(\kappa; m, n_1, n_2) = 0 \quad \text{for } n_1 + n_2 > m \quad (2.10)$$

as a consequence. In addition, there are weight functions $U(\kappa; s, t_1, t_2)$ which are analytic in s at the respective singular point, corresponding to $\mu(\kappa) = 0, 1, 2$ (without any further relation such as (2.5)).

In order to avoid unnecessary complications, we want to assume that the non-trivial exponent μ according to (2.5) is not equal to an integer. This can always be guaranteed by a suitable choice of the still disposable parameter λ .

Since the exponents $\mu(\kappa)$, ν_1 , ν_2 are restricted only by (2.5) but otherwise arbitrary, there are other solutions of the partial differential equation (2.2) relevant as weight functions in our contour integrals (2.1). We may assume that ν_1 and ν_2 are positive integers, preferentially

$$\nu_1 = \nu_2 = 1, \quad (2.11)$$

but for the time being we want to keep ν_1 and ν_2 in the formulas. If ν_1 and ν_2 are increased by any positive integers q_1 and q_2 , respectively, and $\mu(\kappa)$ simultaneously is decreased by $(2/3)q_1 + (1/3)q_2$, then (2.5) is still satisfied. We therefore have to consider, for $q_1, q_2 = 0, 1, 2, \dots$, the set of solutions

$$\begin{aligned} V(\kappa; q_1, q_2) &= V(\kappa; q_1, q_2; s, t_1, t_2) \\ &= \sum_{m=0}^{\infty} \sum_{n_1=0}^m \sum_{n_2=0}^m \Gamma(\nu_1 + q_1 + n_1) \Gamma(\nu_2 + q_2 + n_2) \Gamma(-\mu(\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2 - m) \\ &\quad \times b(\kappa; m, n_1, n_2) (s - \kappa s_0)^{\mu(\kappa) + \frac{2}{3}q_1 + \frac{1}{3}q_2 + m} (t_1 - \kappa t_{10})^{-\nu_1 - q_1 - n_1} (t_2 - \kappa t_{20})^{-\nu_2 - q_2 - n_2}. \end{aligned} \quad (2.12)$$

As indicated, we will use a short-hand notation suppressing the dependence on the variables s, t_1, t_2 . We now have to choose appropriate contours for the integral representation (2.1). For each of the integrals over t_1 or t_2 a closed circle, traversed once in the positive sense, around the relevant singular point of the integrand is appropriate, since the pertinent exponent is an integer. For the s -integral we need an infinite contour which starts somewhere at infinity where the exponential factor of the integral vanishes, surrounds one of the singular points in the positive sense, and returns (on a different sheet) to the starting point. Assuming that s_0 is real and positive, then, if

$$0 < \arg(z) < \frac{1}{3}\pi \quad (2.13)$$

the starting- and end-point at infinity has the phase $\pi/2$. If we agree that

$$\arg(s - \kappa s_0) = 0 \quad (2.14)$$

when s is positive and sufficiently large, the integral of a single term of the infinite series (2.4) can be evaluated:

$$\begin{aligned} & z^\lambda (2\pi i)^{-3} \oint_{\kappa s_0 + i\infty}^{(\kappa t_{20}+)} \oint_{\kappa s_0 + i\infty}^{(\kappa t_{10}+)} \int_{\kappa s_0 + i\infty}^{(\kappa s_0+)} \exp(z^3 s + z^2 t_1 + z t_2) \\ & \times (s - \kappa s_0)^{\mu(\kappa)+m} (t_1 - \kappa t_{10})^{-\nu_1 - n_1} (t_2 - \kappa t_{20})^{-\nu_2 - n_2} ds dt_1 dt_2 \\ & = \exp(\kappa s_0 z^3 + \kappa t_{10} z^2 + \kappa t_{20} z) \frac{z^{-\tau(\kappa) - 3m + 2n_1 + n_2}}{\Gamma(-\mu(\kappa) - m) \Gamma(\nu_1 + n_1) \Gamma(\nu_2 + n_2)}, \end{aligned} \quad (2.15)$$

where, in the power of z on the right-hand side, already use has been made of (2.5). As a consequence, the integral of $V(\kappa; q_1, q_2)$ yields, if the series is integrated term by term, one or the other of the formal solutions (1.2)-(1.3):

$$\begin{aligned} f_{\infty j}(z) &:= z^\lambda (2\pi i)^{-3} \oint_{\kappa s_0 + i\infty}^{(\kappa t_{20}+)} \oint_{\kappa s_0 + i\infty}^{(\kappa t_{10}+)} \int_{\kappa s_0 + i\infty}^{(\kappa s_0+)} \exp(z^3 s + z^2 t_1 + z t_2) \\ & \times V(\kappa; q_1, q_2; s, t_1, t_2) ds dt_1 dt_2 \sim f_{\infty j}^{\text{asy}}(z) \end{aligned} \quad (2.16)$$

in the sector $0 < \arg(z) < \frac{1}{3}\pi$, where $j = 1$ if $\kappa = 1$ or $j = 2$ if $\kappa = -1$. Each of the solutions defined by the integral representation (2.16) has one of the formal solutions (1.2) or (1.3) as its asymptotic expansion as $z \rightarrow \infty$ in the indicated sector. It follows by rotation of the contour that the asymptotic expansions are theoretically valid in the larger sectors $-\frac{1}{6}\pi < \arg(z) < \frac{5}{6}\pi$ for $j = 1$ or $-\frac{1}{2}\pi < \arg(z) < \frac{1}{2}\pi$ for $j = 2$, respectively.

Looking at (2.15) we may see that all the terms for which $3m - 2n_1 - n_2$ is the same yield the same power of z . For the coefficients of the asymptotic expansions (1.2) or (1.3) we then have the representation

$$a_n(\kappa) = \sum_{(m, n_1, n_2) \in I_n} b(\kappa; m, n_1, n_2), \quad (2.17)$$

where

$$I_n = \{(m, n_1, n_2) : 3m - 2n_1 - n_2 = n\} \subset \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0, \quad n = 0, 1, \dots \quad (2.18)$$

Because of the properties of the $b(m, n_1, n_2)$, the sum in (2.18) is finite for each finite n , in particular we have

$$a_0(\kappa) = b_0(\kappa; 0, 0, 0) = 1. \quad (2.19)$$

3. Analytic continuation of the integrand

Below we have to consider the integral representation (2.1) with t_1 and t_2 -contours which are simple closed curves surrounding in the positive sense both the finite singular points $-t_{10}$ and t_{10} or $-t_{20}$ and t_{20} , respectively, and with an s -contour which starts at or near $-s_0 + i\infty$, surrounds both the finite singular points $-s_0$ and s_0 once in the positive sense and ends at $s_0 + i\infty$. We therefore need the analytic continuation of the integrand between the two singular points along this contour. With appropriate power factors Φ included for later convenience, the continuation formula reads

$$\begin{aligned} & \Phi(\kappa; r_1, r_2) V(\kappa; r_1, r_2) \\ &= \sum_{q_1=r_1}^{\infty} \sum_{q_2=r_2}^{\infty} E(-\kappa; r_1, r_2; q_1, q_2) \Phi(-\kappa; q_1, q_2) V(-\kappa; q_1, q_2) + U(-\kappa; r_1, r_2), \end{aligned} \quad (3.1)$$

where

$$\Phi(\kappa; r_1, r_2) = (-2\kappa s_0)^{-\mu(\kappa) - \frac{2}{3}r_1 - \frac{1}{3}r_2}. \quad (3.2)$$

The effect of these power factors is that in (3.1) the total powers with non-integer exponents are powers of $\frac{1}{2}[1 - s/(\kappa s_0)]$ or of $\frac{1}{2}[1 + s/(\kappa s_0)]$, respectively. We may agree that $\arg(1 - s/s_0) = \arg(1 + s/s_0) = 0$ when s is on the real axis between $-s_0$ and s_0 . Also, as above, $\arg(s) = 0$ when s is larger than s_0 . Then, by analytic continuation along the contour under consideration, we have

$$\Phi(1; r_1, r_2) = (2s_0)^{-\mu(1) - \frac{2}{3}r_1 - \frac{1}{3}r_2} \exp(i\pi(\mu(1) + \frac{2}{3}r_1 + \frac{1}{3}r_2)), \quad (3.3)$$

$$\Phi(-1; r_1, r_2) = (2s_0)^{-\mu(-1) - \frac{2}{3}r_1 - \frac{1}{3}r_2}. \quad (3.4)$$

Let us rewrite (3.1) using an even more condensed notation, writing q for (q_1, q_2) in the parameter list of the various functions and writing a sum over q in place of a double sum over q_1 and q_2 , etc. The above continuation formula then reads

$$\Phi(\kappa; r) V(\kappa; r) = \sum_{q=r} E(-\kappa; r, q) \Phi(-\kappa; q) V(-\kappa; q) + U(-\kappa; r), \quad (3.5)$$

and the second continuation formula

$$U(\kappa; r) = \Phi(-\kappa, r)V(-\kappa, r) - \sum_{q=r} \sum_{p=q} E(\kappa; r, q)E(-\kappa; q, p)\Phi(-\kappa; p)V(-\kappa; p) - \sum_{q=r} E(\kappa; r, q)U(-\kappa; q) \quad (3.6)$$

follows by the requirement of consistency of (3.5) for κ replaced by $-\kappa$.

4. Asymptotic expansions for the coefficients in the continuation formula

We now want to determine the E -coefficients in the continuation formulas (3.5) , (3.6) by means of the asymptotic method of Darboux [16] applied to the variable s . The left-hand side of the continuation formula (3.5) is

$$\begin{aligned} & \left(\frac{1}{2} - \frac{s}{2\kappa s_0}\right)^{\mu(\kappa) + \frac{2}{3}r_1 + \frac{1}{3}r_2} \sum_{m=0}^{\infty} \sum_{n_1=0}^m \sum_{n_2=0}^m \Gamma(\nu_1 + r_1 + n_1)\Gamma(\nu_2 + r_2 + n_2) \\ & \times \Gamma(-\mu(\kappa) - \frac{2}{3}r_1 - \frac{1}{3}r_2 - m)b(\kappa; m, n_1, n_2)(-2\kappa s_0)^m \\ & \times \left(\frac{1}{2} - \frac{s}{2\kappa s_0}\right)^m (t_1 - \kappa t_{10})^{-\nu_1 - r_1 - n_1} (t_2 - \kappa t_{20})^{-\nu_2 - r_2 - n_2}. \end{aligned} \quad (4.1)$$

The leading singular term, when $(s, t_1, t_2) \rightarrow (-\kappa s_0, -\kappa t_{10}, -\kappa t_{20})$, on the right-hand side is

$$\begin{aligned} & \sum_{q_1=r_1}^{\infty} \sum_{q_2=r_2}^{\infty} E(-\kappa; r_1, r_2; q_1, q_2)\Gamma(-\mu(-\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2) \\ & \times \Gamma(\nu_1 + q_1)\Gamma(\nu_2 + q_2)b(-\kappa; 0, 0, 0) \\ & \times \left(\frac{1}{2} + \frac{s}{2\kappa s_0}\right)^{\mu(-\kappa) + \frac{2}{3}q_1 + \frac{1}{3}q_2} (t_1 + \kappa t_{10})^{-\nu_1 - q_1} (t_2 + \kappa t_{20})^{-\nu_2 - q_2}. \end{aligned} \quad (4.2)$$

By means of the binomial theorem in its hypergeometric-series-form

$$(1-x)^{-\alpha} = \sum_{j=0}^{\infty} \frac{(\alpha)_j}{j!} x^j, \quad (4.3)$$

where

$$(\alpha)_j = \alpha(\alpha+1)\dots(\alpha+j-1) = \Gamma(\alpha+j)/\Gamma(\alpha)$$

means the Pochhammer symbol, we may expand, if $|t|$ is sufficiently large,

$$(t + \kappa t_0)^{-\nu-q} = \sum_{j=0}^{\infty} \frac{(\nu+q)_j}{j!} (-2\kappa t_0)^j (t - \kappa t_0)^{-\nu-q-j} \quad (4.4)$$

and, if $|s/(\kappa s_0)|$ is sufficiently small,

$$\left(\frac{1}{2} + \frac{s}{2\kappa s_0}\right)^{\mu(-\kappa) + \frac{2}{3}q_1 + \frac{1}{3}q_2} = \sum_{m=0}^{\infty} \frac{(-\mu(-\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2)_m}{m!} \left(\frac{1}{2} - \frac{s}{2\kappa s_0}\right)^m. \quad (4.5)$$

Then the leading singular term on the right becomes

$$\begin{aligned} & \sum_{q_1=r_1}^{\infty} \sum_{q_2=r_2}^{\infty} E(-\kappa; r_1, r_2; q_1, q_2) \Gamma(-\mu(-\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2) \\ & \times \Gamma(\nu_1 + q_1) \sum_{j_1=0}^{\infty} \frac{(\nu_1 + q_1)_{j_1}}{j_1!} (-2\kappa t_{10})^{j_1} (t_1 - \kappa t_{10})^{-\nu_1 - q_1 - j_1} \\ & \times \Gamma(\nu_2 + q_2) \sum_{j_2=0}^{\infty} \frac{(\nu_2 + q_2)_{j_2}}{j_2!} (-2\kappa t_{20})^{j_2} (t_2 - \kappa t_{20})^{-\nu_2 - q_2 - j_2} \\ & \times b(-\kappa; 0, 0, 0) \sum_{m=0}^{\infty} \frac{(-\mu(-\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2)_m}{m!} \left(\frac{1}{2} - \frac{s}{2\kappa s_0}\right)^m. \end{aligned} \quad (4.6)$$

The coefficients of this series should agree asymptotically, as $m \rightarrow \infty$, with those of the series on the left-hand side, where the power factor in front of the series (4.1), when $s \rightarrow -\kappa s_0$, tends to 1 and may be omitted. We therefore obtain

$$\begin{aligned} & \sum_{n_1=0}^m \sum_{n_2=0}^m \Gamma(\nu_1 + r_1 + n_1) \Gamma(\nu_2 + r_2 + n_2) \Gamma(-\mu(\kappa) - \frac{2}{3}r_1 - \frac{1}{3}r_2 - m) \\ & \times b(\kappa; m, n_1, n_2) (-2\kappa s_0)^m (t_1 - \kappa t_{10})^{-\nu_1 - r_1 - n_1} (t_2 - \kappa t_{20})^{-\nu_2 - r_2 - n_2} \\ & \approx \sum_{q_1=r_1}^{\infty} \sum_{q_2=r_2}^{\infty} E(-\kappa; r_1, r_2; q_1, q_2) \Gamma(-\mu(-\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2) \\ & \times \Gamma(\nu_1 + q_1) \sum_{j_1=0}^{\infty} \frac{(\nu_1 + q_1)_{j_1}}{j_1!} (-2\kappa t_{10})^{j_1} (t_1 - \kappa t_{10})^{-\nu_1 - q_1 - j_1} \\ & \times \Gamma(\nu_2 + q_2) \sum_{j_2=0}^{\infty} \frac{(\nu_2 + q_2)_{j_2}}{j_2!} (-2\kappa t_{20})^{j_2} (t_2 - \kappa t_{20})^{-\nu_2 - q_2 - j_2} \\ & \times b(-\kappa; 0, 0, 0) \frac{(-\mu(-\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2)_m}{m!}, \end{aligned} \quad (4.7)$$

which holds asymptotically as $m \rightarrow \infty$. On both sides of this asymptotic equation a double power series in the same variables appears, so the coefficients of the corresponding terms

must be equal. This yields, for each set r_1, r_2, n_1, n_2 ,

$$\begin{aligned} & \Gamma(-\mu(\kappa) - \frac{2}{3}r_1 - \frac{1}{3}r_2 - m)(-2\kappa s_0)^m b(\kappa; m, n_1, n_2) \\ & \approx \sum_{q_1=r_1}^{r_1+n_1} \sum_{q_2=r_2}^{r_2+n_2} E(-\kappa; r_1, r_2; q_1, q_2) \frac{\Gamma(-\mu(-\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2 + m)}{m!} \\ & \times \left(\frac{1}{j_1!} (-2\kappa t_{10})^{j_1} \right)_{j_1=r_1+n_1-q_1} \left(\frac{1}{j_2!} (-2\kappa t_{20})^{j_2} \right)_{j_2=r_2+n_2-q_2} b(-\kappa; 0, 0, 0) \end{aligned} \quad (4.8)$$

or, if we introduce new indices of summation and make use of the reflection formula of the gamma function,

$$\begin{aligned} & -\pi[\Gamma(1+m)]^{-1}(2\kappa s_0)^m b(\kappa; m, n_1, n_2) \\ & \approx \sum_{p_1=0}^{n_1} \sum_{p_2=0}^{n_2} E(-\kappa; r_1, r_2; r_1 + p_1, r_2 + p_2) \frac{\Gamma(-\mu(-\kappa) - \frac{2}{3}r_1 - \frac{1}{3}r_2 - \frac{2}{3}p_1 - \frac{1}{3}p_2 + m)}{\Gamma(1+m)} \\ & \times \frac{\Gamma(1 + \mu(\kappa) + \frac{2}{3}r_1 + \frac{1}{3}r_2 + m)}{\Gamma(1+m)} \sin(\pi[\mu(\kappa) + \frac{2}{3}r_1 + \frac{1}{3}r_2]) \\ & \times \left(\frac{1}{j_1!} (-2\kappa t_{10})^{j_1} \right)_{j_1=n_1-p_1} \left(\frac{1}{j_2!} (-2\kappa t_{20})^{j_2} \right)_{j_2=n_2-p_2} b(-\kappa; 0, 0, 0). \end{aligned} \quad (4.9)$$

Now the left-hand side is independent of r_1 and r_2 and so is the product of the two ratios of gamma functions, when $m \rightarrow \infty$, on the right. Therefore $E(-\kappa; r_1, r_2; r_1 + p_1, r_2 + p_2) \sin(\pi[\mu(\kappa) + \frac{2}{3}r_1 + \frac{1}{3}r_2])$ must be independent of r_1 and r_2 too. This proves

Lemma 3.

$$E(-\kappa; r_1, r_2; r_1 + p_1, r_2 + p_2) = \frac{\sin(\pi\mu(\kappa))}{\sin(\pi[\mu(\kappa) + \frac{2}{3}r_1 + \frac{1}{3}r_2])} E(-\kappa; 0, 0; p_1, p_2). \quad (4.10)$$

After Lemma 3 is available, we need to determine the E -coefficients for $r_1 = r_2 = 0$ only, and it is advantageous to introduce the closely related coefficients

$$e(-\kappa; p_1, p_2) = \sin(\pi\mu(\kappa)) E(-\kappa; 0, 0; p_1, p_2), \quad (4.11)$$

which are independent of λ . Eq (4.9), with $r_1 = r_2 = 0$ and the factor $b(-\kappa; 0, 0, 0)$, which is equal to 1, omitted, then becomes

$$\begin{aligned} & \frac{-\pi}{\Gamma(1 + \mu(\kappa) + m)} (2\kappa s_0)^m b(\kappa; m, n_1, n_2) \\ & \approx \sum_{p_1=0}^{n_1} \sum_{p_2=0}^{n_2} e(-\kappa; p_1, p_2) \frac{\Gamma(-\mu(-\kappa) - \frac{2}{3}p_1 - \frac{1}{3}p_2 + m)}{\Gamma(1+m)} \\ & \times \left(\frac{1}{j_1!} (-2\kappa t_{10})^{j_1} \right)_{j_1=n_1-p_1} \left(\frac{1}{j_2!} (-2\kappa t_{20})^{j_2} \right)_{j_2=n_2-p_2}. \end{aligned} \quad (4.12)$$

We then may solve (4.12) for the e -coefficient with $p_1 = n_1, p_2 = n_2$ and obtain the following asymptotic formula in terms of a b -coefficient and the earlier e -coefficients,

$$\begin{aligned}
e(-\kappa; n_1, n_2) &\approx -\pi(2\kappa s_0)^m \\
&\times \frac{\Gamma(1+m)}{\Gamma(1+\mu(\kappa)+m)\Gamma(-\mu(-\kappa)-\frac{2}{3}n_1-\frac{1}{3}n_2+m)} b(\kappa; m, n_1, n_2) \\
&- \sum_{\substack{p_1=0 \\ (p_1, p_2) \neq (n_1, n_2)}}^{n_1} \sum_{p_2=0}^{n_2} e(-\kappa; p_1, p_2) \frac{\Gamma(-\mu(-\kappa)-\frac{2}{3}p_1-\frac{1}{3}p_2+m)}{\Gamma(-\mu(-\kappa)-\frac{2}{3}n_1-\frac{1}{3}n_2+m)} \\
&\times \left(\frac{1}{j_1!} (-2\kappa t_{10})^{j_1} \right)_{j_1=n_1-p_1} \left(\frac{1}{j_2!} (-2\kappa t_{20})^{j_2} \right)_{j_2=n_2-p_2}.
\end{aligned} \tag{4.13}$$

By repeated application of this formula, all the e -coefficients on the right-hand side may be eliminated, and this yields the remarkable explicit limit formula

$$\begin{aligned}
e(-\kappa; n_1, n_2) &= -\pi \\
&\times \lim_{m \rightarrow \infty} \frac{\Gamma(1+m)}{\Gamma(1+\mu(\kappa)+m)\Gamma(-\mu(-\kappa)-\frac{2}{3}n_1-\frac{1}{3}n_2+m)} \\
&\times (2\kappa s_0)^m \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} b(\kappa; m, j_1, j_2) \frac{(2\kappa t_{10})^{n_1-j_1}}{(n_1-j_1)!} \frac{(2\kappa t_{20})^{n_2-j_2}}{(n_2-j_2)!}.
\end{aligned} \tag{4.14}$$

The proof of this formula will be given below in Section 7.

Although approximate asymptotic equations or limit formulas such as (4.13) or (4.14) are interesting from a theoretical point of view, we finally need more, namely a detailed asymptotic expansion suitable for accurate numerical evaluation. This can be obtained, on the basis of Schäfke and Schmidt [17], essentially in the same way as above, apart from the following two refinements: The power factor in front of (4.1) can no longer be omitted, and we have to include a finite number of singular terms rather than the leading one, (4.2), alone. The result of this procedure, which will be derived in more detail below, is

Theorem 1. *The E -coefficients, or e -coefficients according to (4.11), in the continuation*

formula (3.1) or (3.5) are

$$\begin{aligned}
e(-\kappa; n_1, n_2) &= -\pi \\
&\times \left[\frac{\Gamma(1+m)}{\Gamma(1+\mu(\kappa)+m)\Gamma(-\mu(-\kappa)-\frac{2}{3}n_1-\frac{1}{3}n_2+m)} (2\kappa s_0)^m b(\kappa; m, n_1, n_2) \right. \\
&- \sum_{\substack{q_1=0 \\ (q_1, q_2) \neq (n_1, n_2)}}^{n_1} \sum_{q_2=0}^{n_2} e(-\kappa; q_1, q_2) \frac{\Gamma(-\mu(-\kappa)-\frac{2}{3}q_1-\frac{1}{3}q_2+m)}{\Gamma(-\mu(-\kappa)-\frac{2}{3}n_1-\frac{1}{3}n_2+m)} \\
&\times [1 + \sum_{k=1}^K \sum_{l_1=0}^k \sum_{l_2=0}^k \frac{(1+\mu(-\kappa)+\frac{2}{3}q_1+\frac{1}{3}q_2)_k}{(1+\mu(-\kappa)+\frac{2}{3}q_1+\frac{1}{3}q_2-m)_k} H(-\kappa; k, l_1, l_2; q_1, q_2) + O(m^{-K-1})] \\
&\times \left(\frac{1}{j_1!} (-2\kappa t_{10})^{j_1} \right)_{j_1+q_1+l_1=n_1} \left(\frac{1}{j_2!} (-2\kappa t_{20})^{j_2} \right)_{j_2+q_2+l_2=n_2} \Big] \\
&\times \left[1 + \sum_{k=1}^K \frac{(1+\mu(-\kappa)+\frac{2}{3}n_1+\frac{1}{3}n_2)_k}{(1+\mu(-\kappa)+\frac{2}{3}n_1+\frac{1}{3}n_2-m)_k} H(-\kappa; k, 0, 0; n_1, n_2) + O(m^{-K-1}) \right]^{-1},
\end{aligned} \tag{4.15}$$

where

$$H(-\kappa; k, l_1, l_2; q_1, q_2) = \sum_{j=l_1+l_2}^k \frac{(\mu(\kappa))_{k-j}}{(k-j)!(1+\mu(-\kappa)+\frac{2}{3}q_1+\frac{1}{3}q_2)_j} (-2\kappa s_0)^j b(-\kappa; j, l_1, l_2). \tag{4.16}$$

With a suitable choice of m and K , Theorem 1 may be used to compute accurate values of the e -coefficients, beginning with

$$\begin{aligned}
e(-\kappa; 0, 0) &= -\pi \frac{\Gamma(1+m)}{\Gamma(1+\mu(\kappa)+m)\Gamma(-\mu(-\kappa)+m)} (2\kappa s_0)^m b(\kappa; m, 0, 0) \\
&\times \left[1 + \sum_{k=1}^K \frac{(1+\mu(-\kappa))_k}{(1+\mu(-\kappa)-m)_k} H(-\kappa; k, 0, 0; 0, 0) + O(m^{-K-1}) \right]^{-1}.
\end{aligned} \tag{4.17}$$

While the e -coefficients are independent of λ , the approximate values computed for any finite m do depend on it. Thus λ here plays the role of a computational parameter which could be adjusted for optimal accuracy. The best choice from this point of view, in particular when L is not small, is $\lambda = L$ or $\lambda = -L$, according to the discussion in Section 7 below.

5. Multiplicative solutions and the Floquet exponent

We now want to construct a linear combination of the solutions at infinity,

$$f_p(z) = \alpha f_{\infty 1}(z) + \beta f_{\infty 2}(z), \quad (5.1)$$

which is a multiplicative solution such that, after analytic continuation along a sufficiently large circle around the origin traversed once in the negative sense, this solution remains the same apart from multiplication by a constant factor, that is

$$f_p(e^{-2\pi i} z) = p f_p(z). \quad (5.2)$$

This solution is proportional to one of the Floquet solutions introduced above in the introduction.

We use the integral representation with contours which surround both the finite singular points as introduced above in Section 3. This integral is equal to the sum of the two integrals with contours, also considered above in Section 2, which surround only one of these singular points. If the contour is kept fixed, then the circle in the z -plane traversed once in the negative sense corresponds to a circle, around the origin and with radius greater than s_0 , in the s -plane traversed three times in the positive sense, since then $z^3 s$ in the exponential part of the integrand does not change.

Let us consider the integral representation with $V(1; 0)$ in the integrand and with the above phase conventions. From (3.5) we have

$$\Phi(1; 0)V(1; 0) = \sum_{q=0} E(-1; 0, q)\Phi(-1; q)V(-1; q) + U(-1; 0). \quad (5.3)$$

The integral then yields a function

$$f^I(z) := \Phi(1; 0)f_{\infty 1}(z) + \sum_{q=0} E(-1; 0, q)\Phi(-1; q)f_{\infty 2}(z), \quad (5.4)$$

where the two terms come from the two singular points, which contribute via the left- or right-hand side of (5.3), respectively. Let us now consider, in the s -plane, a simple closed loop, homotopic to the circle mentioned above, consisting of small circles around the singular points s_0 and $-s_0$ and straight line segments along the real axis between these singular points. Let us start with the continuation formula (5.3) in a neighbourhood of

the origin, where the left- as well as the right-hand side of (5.3) are valid, and see what happens when we follow the loop in the positive sense. We shall always refer to the above phase conventions and display any additional phases explicitly. Traversing the circular part around s_0 multiplies the function $V(1;0)$ by the phase factor $\exp(2\pi i\mu(1))$, so that we then have

$$\begin{aligned} & \exp(2\pi i\mu(1))\Phi(1;0)V(1;0) \\ &= \exp(2\pi i\mu(1))\left\{\sum_{q=0} E(-1;0,q)\Phi(-1;q)V(-1;q) + U(-1;0)\right\}, \end{aligned} \quad (5.5)$$

where the right-hand side is the analytic continuation of the left by means of (5.3). Next we have to traverse the circular part around $-s_0$, which multiplies the function $V(-1;q)$ by the phase factor $\exp(2\pi i[\mu(-1) + \varphi(q)])$, where

$$\varphi(q) = \frac{2}{3}q_1 + \frac{1}{3}q_2, \quad (5.6)$$

so that we have on the straight line segment from $-s_0$ to s_0 , after the loop has been traversed once,

$$\begin{aligned} & \exp(2\pi i[\mu(1) + \mu(-1)]) \sum_{q=0} E(-1;0,q) \exp(2\pi i\varphi(q))\Phi(-1;q)V(-1;q) \\ &+ \exp(2\pi i\mu(1))U(-1;0) \\ &= \exp(2\pi i\mu(1))\{\Phi(1;0)V(1;0) + \sum_{q=0} \{\exp(2\pi i[\mu(-1) + \varphi(q)]) - 1\}U(1;q) \\ &+ \sum_{q=0} \sum_{p=q} E(-1;0,q)\{\exp(2\pi i[\mu(-1) + \varphi(q)]) - 1\}E(1;q,p)\Phi(1;p)V(1;p)\}, \end{aligned} \quad (5.7)$$

where again the right-hand side is the analytic continuation of the left. It is evident that following the loop further leads to increasingly lengthier and more complicated formulas, not suitable for being fully displayed in this paper. We therefore want to stop here for a moment and consider the integral representation with the integrand obtained after the loop in the s -plane has been traversed only once (rather than three times, as finally needed). A representative example of the terms in (5.7) then is

$$\sum_{q=0} \sum_{p=q} E(-1;0,q)\{\exp(2\pi i[\mu(-1) + \varphi(q)]) - 1\}E(1;q,p)\Phi(1;p)V(1;p). \quad (5.8)$$

The integral of $V(1; p)$ yields $f_{\infty 1}(z)$, independent of p , so that the sum over p can now be performed and, because of Lemma 3, the multiple sum reduces to a product of single sums. After integration we therefore obtain for (5.8)

$$\begin{aligned} & \{2i \sin(\pi\mu(-1)) \exp(i\pi\mu(-1)) \sum_{q=0}^{\infty} E(-1; 0, q)(2s_0)^{-\varphi(q)} \exp(2\pi i\varphi(q)) \\ & \times \sum_{\tilde{p}=0}^{\infty} E(1; 0, \tilde{p})(2s_0)^{-\varphi(\tilde{p})} \exp(i\pi\varphi(\tilde{p}))\} \Phi(1; 0) f_{\infty 1}(z). \end{aligned} \quad (5.9)$$

It is now convenient to introduce the phase factor

$$\eta := \exp(\tfrac{1}{3}\pi i) = \tfrac{1}{2}(1 + i\sqrt{3}) \quad (5.10)$$

which satisfies

$$\eta^6 = 1, \quad 1 + \eta^2 + \eta^4 = 0. \quad (5.11)$$

Also, for the sums occurring here and below we may introduce the "Stokes multipliers"

$$\begin{aligned} \sigma_n(1) &:= \sin(\pi\mu(1)) \sum_{q=0}^{\infty} E(-1; 0, q)(2s_0)^{-\varphi(q)} \exp(2n\pi i\varphi(q)), \\ \sigma_n(-1) &:= \sin(\pi\mu(-1)) \sum_{q=0}^{\infty} E(1; 0, q)(2s_0)^{-\varphi(q)} \exp((2n+1)\pi i\varphi(q)), \end{aligned} \quad (5.12)$$

which become

$$\begin{aligned} \sigma_0(1) &= S_0(1) + (2s_0)^{-1/3} S_1(1) + (2s_0)^{-2/3} S_2(1), \\ \sigma_0(-1) &= S_0(-1) + \eta(2s_0)^{-1/3} S_1(-1) + \eta^2(2s_0)^{-2/3} S_2(-1), \\ \sigma_1(1) &= S_0(1) + \eta^2(2s_0)^{-1/3} S_1(1) + \eta^4(2s_0)^{-2/3} S_2(1), \\ \sigma_1(-1) &= S_0(-1) + \eta^3(2s_0)^{-1/3} S_1(-1) + (2s_0)^{-2/3} S_2(-1), \\ \sigma_2(1) &= S_0(1) + \eta^4(2s_0)^{-1/3} S_1(1) + \eta^2(2s_0)^{-2/3} S_2(1), \\ \sigma_2(-1) &= S_0(-1) + \eta^5(2s_0)^{-1/3} S_1(-1) + \eta^4(2s_0)^{-2/3} S_2(-1) \end{aligned} \quad (5.13)$$

in terms of the real partial sums

$$\begin{aligned} S_0(\kappa) &:= \sum_{l=0}^{\infty} (2\kappa s_0)^{-l} \sum_{(n_1, n_2) \in J_{3l}} e(-\kappa; n_1, n_2) = e(-\kappa; 0, 0) + \dots, \\ S_1(\kappa) &:= \sum_{l=0}^{\infty} (2\kappa s_0)^{-l} \sum_{(n_1, n_2) \in J_{3l+1}} e(-\kappa; n_1, n_2) = e(-\kappa; 0, 1) + \dots, \\ S_2(\kappa) &:= \sum_{l=0}^{\infty} (2\kappa s_0)^{-l} \sum_{(n_1, n_2) \in J_{3l+2}} e(-\kappa; n_1, n_2) = e(-\kappa; 1, 0) + e(-\kappa; 0, 2) + \dots, \end{aligned} \quad (5.14)$$

where

$$J_l = \{(n_1, n_2) : 2n_1 + n_2 = l\} \subset \mathbb{N}_0 \times \mathbb{N}_0, \quad l = 0, 1, \dots \quad (5.15)$$

The integral of the representative term (5.9) then becomes

$$\frac{2i \exp(i\pi\mu(-1))}{\sin(\pi\mu(1))} \sigma_1(1) \sigma_0(-1) \Phi(1; 0) f_{\infty 1}(z). \quad (5.16)$$

In total, we have

$$\begin{aligned} \exp(2\pi i \lambda / 3) f^I(e^{-2\pi i / 3} z) &= \frac{\exp(2\pi i [\mu(1) + \mu(-1)])}{\sin(\pi\mu(1))} \sigma_1(1) \Phi(-1; 0) f_{\infty 2}(z) \\ &+ \exp(2\pi i \mu(1)) \left\{ 1 + \frac{2i \exp(\pi i \mu(-1))}{\sin(\pi\mu(1))} \sigma_1(1) \sigma_0(-1) \right\} \Phi(1; 0) f_{\infty 1}(z), \end{aligned} \quad (5.17)$$

where

$$f^I(z) = \Phi(1; 0) f_{\infty 1}(z) + \frac{1}{\sin(\pi\mu(1))} \sigma_0(1) \Phi(-1; 0) f_{\infty 2}(z). \quad (5.18)$$

So far we have traversed the s -loop once and obtained on it the analytic continuation of the integrand, but we have to traverse it three times. This yields

$$f^I(e^{-2\pi i} z) = S_{11} \Phi(1; 0) f_{\infty 1}(z) + S_{12} \Phi(-1; 0) f_{\infty 2}(z) \quad (5.19)$$

with lengthy expressions for S_{11} and S_{12} .

In a similar way, the integral representation with $V(-1; 0)$ yields

$$f^{\text{II}}(z) := \sum_{q=0} E(1; 0, q) \Phi(1; q) f_{\infty 1}(z) + \Phi(-1; 0) f_{\infty 2}(z) \quad (5.20)$$

or

$$f^{\text{II}}(z) = \frac{1}{\sin(\pi\mu(-1))} \sigma_0(-1) \Phi(1; 0) f_{\infty 1}(z) + \Phi(-1; 0) f_{\infty 2}(z) \quad (5.21)$$

and leads, after the loop has been traversed once, to

$$\begin{aligned} \exp(2\pi i \lambda / 3) f^{\text{II}}(e^{-2\pi i / 3} z) &= \left\{ 2i \exp(\pi i \mu(-1)) \sigma_0(-1) + \frac{\exp(2\pi i \mu(1))}{\sin(\pi\mu(-1))} \sigma_1(-1) \right. \\ &- \frac{4 \exp(\pi i [\mu(1) + \mu(-1)])}{\sin(\pi\mu(-1))} \sigma_1(-1) \sigma_1(1) \sigma_0(-1) \left. \right\} \Phi(1; 0) f_{\infty 1}(z) \\ &+ \exp(2\pi i \mu(-1)) \left\{ 1 + \frac{2i \exp(\pi i \mu(1))}{\sin(\pi\mu(-1))} \sigma_1(-1) \sigma_1(1) \right\} \Phi(-1; 0) f_{\infty 2}(z) \end{aligned} \quad (5.22)$$

and, after three times,

$$f^{\text{II}}(e^{-2\pi i}z) = S_{21}\Phi(1;0)f_{\infty 1}(z) + S_{22}\Phi(-1;0)f_{\infty 2}(z), \quad (5.23)$$

where again the expressions for S_{21} and S_{22} are too lengthy to be displayed here.

What we really want to obtain are the circuit relations for $f_{\infty 1}$ and $f_{\infty 2}$, that is

$$\begin{aligned} f_{\infty 1}(e^{-2\pi i}z) &= T_{11}f_{\infty 1}(z) + T_{12}f_{\infty 2}(z), \\ f_{\infty 2}(e^{-2\pi i}z) &= T_{21}f_{\infty 1}(z) + T_{22}f_{\infty 2}(z), \end{aligned} \quad (5.24)$$

where

$$\begin{aligned} T_{11} &= [S_{11} - \frac{\sigma_0(1)}{\sin(\pi\mu(1))}S_{21}]/[1 - \frac{\sigma_0(-1)\sigma_0(1)}{\sin(\pi\mu(-1))\sin(\pi\mu(1))}], \\ T_{12} &= [\Phi(-1;0)/\Phi(1;0)][S_{12} - \frac{\sigma_0(1)}{\sin(\pi\mu(1))}S_{22}]/[1 - \frac{\sigma_0(-1)\sigma_0(1)}{\sin(\pi\mu(-1))\sin(\pi\mu(1))}], \\ T_{21} &= [\Phi(1;0)/\Phi(-1;0)][S_{21} - \frac{\sigma_0(-1)}{\sin(\pi\mu(-1))}S_{11}]/[1 - \frac{\sigma_0(-1)\sigma_0(1)}{\sin(\pi\mu(-1))\sin(\pi\mu(1))}], \\ T_{22} &= [S_{22} - \frac{\sigma_0(-1)}{\sin(\pi\mu(-1))}S_{12}]/[1 - \frac{\sigma_0(-1)\sigma_0(1)}{\sin(\pi\mu(-1))\sin(\pi\mu(1))}]. \end{aligned} \quad (5.25)$$

It turns out that each numerator contains a common factor which compensates the denominator. The result then is

Theorem 2. *The coefficients in the circuit relations (5.24) are*

$$\begin{aligned} T_{11} &= \exp(2\pi i\tau(1)) \\ &+ 4\exp(\frac{4}{3}\pi i\tau(-1))\sigma_0(-1)\sigma_0(1) + 4\exp(\frac{4}{3}\pi i\tau(1))\sigma_1(1)\sigma_0(-1) + 4\sigma_2(1)\sigma_0(-1) \\ &+ 4\exp(\frac{4}{3}\pi i\tau(1))\sigma_2(1)\sigma_1(-1) + 4\exp(\frac{4}{3}\pi i\tau(1))\sigma_2(-1)\sigma_0(1) + 4\sigma_1(-1)\sigma_0(1) \\ &+ 16\exp(\frac{2}{3}\pi i\tau(1))\sigma_2(-1)\sigma_2(1)\sigma_1(-1)\sigma_0(1) + 16\exp(\frac{2}{3}\pi i\tau(1))\sigma_2(-1)\sigma_1(1)\sigma_0(-1)\sigma_0(1) \\ &+ 16\exp(\frac{2}{3}\pi i\tau(1))\sigma_2(1)\sigma_1(-1)\sigma_1(1)\sigma_0(-1) + 16\exp(\frac{2}{3}\pi i\tau(-1))\sigma_2(-1)\sigma_2(1)\sigma_0(-1)\sigma_0(1) \\ &+ 16\exp(\frac{2}{3}\pi i\tau(-1))\sigma_1(-1)\sigma_1(1)\sigma_0(-1)\sigma_0(1) \\ &+ 64\sigma_2(-1)\sigma_2(1)\sigma_1(-1)\sigma_1(1)\sigma_0(-1)\sigma_0(1), \end{aligned} \quad (5.26a)$$

$$\begin{aligned} T_{12} &= (2s_0)^{(1/3)[\tau(1)-\tau(-1)]}\{2i\exp(\frac{2}{3}\pi i\tau(-1))\sigma_2(1) \\ &+ 2i\exp(\frac{2}{3}\pi i\tau(1))\sigma_1(1) + 2i\exp(2\pi i\tau(-1))\sigma_0(1) \\ &+ 8i\exp(\frac{4}{3}\pi i\tau(-1))\sigma_2(-1)\sigma_2(1)\sigma_0(1) + 8i\sigma_2(-1)\sigma_1(1)\sigma_0(1) \\ &+ 8i\exp(\frac{4}{3}\pi i\tau(-1))\sigma_1(-1)\sigma_1(1)\sigma_0(1) + 8i\sigma_2(1)\sigma_1(-1)\sigma_1(1) \\ &+ 32i\exp(\frac{2}{3}\pi i\tau(-1))\sigma_2(-1)\sigma_2(1)\sigma_1(-1)\sigma_1(1)\sigma_0(1)\}, \end{aligned} \quad (5.26b)$$

$$\begin{aligned}
T_{21} = & (2s_0)^{(1/3)[\tau(-1)-\tau(1)]} \{-2i \exp(\frac{4}{3}\pi i \tau(-1))\sigma_0(-1) \\
& - 2i \exp(\frac{4}{3}\pi i \tau(1))\sigma_2(-1) - 2i\sigma_1(-1) \\
& - 8i \exp(\frac{2}{3}\pi i \tau(1))\sigma_2(-1)\sigma_2(1)\sigma_1(-1) \\
& - 8i \exp(\frac{2}{3}\pi i \tau(-1))\sigma_2(-1)\sigma_2(1)\sigma_0(-1) \\
& - 8i \exp(\frac{2}{3}\pi i \tau(-1))\sigma_1(-1)\sigma_1(1)\sigma_0(-1) \\
& - 8i \exp(\frac{2}{3}\pi i \tau(1))\sigma_2(-1)\sigma_1(1)\sigma_0(-1) \\
& - 32i\sigma_2(-1)\sigma_2(1)\sigma_1(-1)\sigma_1(1)\sigma_0(-1)\},
\end{aligned} \tag{5.26c}$$

$$\begin{aligned}
T_{22} = & \exp(2\pi i \tau(-1)) \\
& + 4\exp(\frac{4}{3}\pi i \tau(-1))\sigma_2(-1)\sigma_2(1) + 4\exp(\frac{4}{3}\pi i \tau(-1))\sigma_1(-1)\sigma_1(1) + 4\sigma_2(-1)\sigma_1(1) \\
& + 16\exp(\frac{2}{3}\pi i \tau(-1))\sigma_2(-1)\sigma_2(1)\sigma_1(-1)\sigma_1(1).
\end{aligned} \tag{5.26d}$$

It turns out that

$$T_{11}T_{22} - T_{12}T_{21} = 1. \tag{5.27}$$

We are looking for a multiplicative solution such that (5.2) holds. It then follows from (5.1), (5.2), (5.24) that

$$\begin{aligned}
(T_{11} - p)\alpha + T_{21}\beta &= 0, \\
T_{12}\alpha + (T_{22} - p)\beta &= 0
\end{aligned} \tag{5.28}$$

and, as a consequence, that

$$(T_{11} - p)(T_{22} - p) = T_{12}T_{21} \tag{5.29}$$

or

$$p^2 - (T_{11} + T_{22})p + 1 = 0, \tag{5.30}$$

where (5.27) has been used. The roots p_1, p_2 of this equation satisfy

$$p_1 + p_2 = T_{11} + T_{22}, \tag{5.31}$$

and they may be represented in terms of one (not necessarily real) parameter ω as

$$p_1 = \exp(-2\pi i \omega), \quad p_2 = \exp(2\pi i \omega). \tag{5.32}$$

Then we have

$$p_1 + p_2 = 2 \cos(2\pi\omega) = T_{11} + T_{22}, \quad (5.33)$$

and the final result is

Theorem 3. *The characteristic exponent ω of the multiplicative solutions is given by*

$$\cos(2\pi\omega) = \cos(2\pi\tau(1)) + X, \quad (5.34)$$

where

$$\begin{aligned} X = & 2 \exp\left(\frac{4}{3}\pi i\tau(-1)\right)\sigma_0(-1)\sigma_0(1) + 2 \exp\left(\frac{4}{3}\pi i\tau(1)\right)\sigma_1(1)\sigma_0(-1) + 2\sigma_2(1)\sigma_0(-1) \\ & + 2 \exp\left(\frac{4}{3}\pi i\tau(1)\right)\sigma_2(1)\sigma_1(-1) + 2 \exp\left(\frac{4}{3}\pi i\tau(1)\right)\sigma_2(-1)\sigma_0(1) + 2\sigma_1(-1)\sigma_0(1) \\ & + 2 \exp\left(\frac{4}{3}\pi i\tau(-1)\right)\sigma_2(-1)\sigma_2(1) + 2 \exp\left(\frac{4}{3}\pi i\tau(-1)\right)\sigma_1(-1)\sigma_1(1) + 2\sigma_2(-1)\sigma_1(1) \\ & + 8 \exp\left(\frac{2}{3}\pi i\tau(-1)\right)\sigma_2(-1)\sigma_2(1)\sigma_1(-1)\sigma_1(1) \\ & + 8 \exp\left(\frac{2}{3}\pi i\tau(1)\right)\sigma_2(-1)\sigma_2(1)\sigma_1(-1)\sigma_0(1) + 8 \exp\left(\frac{2}{3}\pi i\tau(1)\right)\sigma_2(-1)\sigma_1(1)\sigma_0(-1)\sigma_0(1) \\ & + 8 \exp\left(\frac{2}{3}\pi i\tau(1)\right)\sigma_2(1)\sigma_1(-1)\sigma_1(1)\sigma_0(-1) + 8 \exp\left(\frac{2}{3}\pi i\tau(-1)\right)\sigma_2(-1)\sigma_2(1)\sigma_0(-1)\sigma_0(1) \\ & + 8 \exp\left(\frac{2}{3}\pi i\tau(-1)\right)\sigma_1(-1)\sigma_1(1)\sigma_0(-1)\sigma_0(1) \\ & + 32\sigma_2(-1)\sigma_2(1)\sigma_1(-1)\sigma_1(1)\sigma_0(-1)\sigma_0(1). \end{aligned} \quad (5.35)$$

For each of the roots p_1 and p_2 we may determine the ratio of α and β from the upper or lower equation of (5.28). Choosing in each case a convenient normalization, we get the desired multiplicative solutions

$$\begin{aligned} f_{p1}(z) &= (T_{22} - p_1)f_{\infty 1}(z) - T_{12}f_{\infty 2}(z), \\ f_{p2}(z) &= (T_{22} - p_2)f_{\infty 1}(z) - T_{12}f_{\infty 2}(z). \end{aligned} \quad (5.36)$$

Here the lower equation of (5.28) has been used in both cases so that T_{11} , which consists of a considerably longer expression than T_{22} , does not appear.

Whenever $D_m = 0$ for all the $m = 1, 2, \dots, 6$, the origin is a regular singular point of the differential equation with exponents $\omega = L, -L$. We are not able, however, to see analytically that then the lengthy expression (5.35) reduces to $X = \cos(2\pi L) - \cos(2\pi\tau(1))$, but this is confirmed in examples of numerical computations, as expected.

6. Asymptotic behaviour of the late coefficients of the formal power series solutions

According to (4.12), the leading terms of the asymptotic behaviour of the b -coefficients for large m are given by

$$\begin{aligned} b(\kappa; m, n_1, n_2) &\approx -\frac{1}{\pi}(2\kappa s_0)^{-m} \frac{\Gamma(1 + \mu(\kappa) + m)}{\Gamma(1 + m)} \\ &\times \sum_{p_1=0}^{n_1} \sum_{p_2=0}^{n_2} e(-\kappa; p_1, p_2) \Gamma(-\mu(-\kappa) - \frac{2}{3}p_1 - \frac{1}{3}p_2 + m) \\ &\times \frac{1}{(n_1 - p_1)!} (-2\kappa t_{10})^{n_1 - p_1} \frac{1}{(n_2 - p_2)!} (-2\kappa t_{20})^{n_2 - p_2}. \end{aligned} \quad (6.1)$$

This result may be used to discuss the asymptotic behaviour of the late a -coefficients of the formal solutions. Writing the decomposition (2.17) of the a -coefficients in terms of the b -coefficients separately for each of three consecutive indices $n = 3N, 3N + 1, 3N + 2$, we have

$$a_{3N}(\kappa) = \sum_{l=0}^{2N} \sum_{(n_1, n_2) \in J_{3l}} b(\kappa; N + l, n_1, n_2) = b(\kappa; N, 0, 0) + \dots, \quad (6.2)$$

$$a_{3N+1}(\kappa) = \sum_{l=1}^{2N+1} \sum_{(n_1, n_2) \in J_{3l-1}} b(\kappa; N + l, n_1, n_2) = b(\kappa; N + 1, 1, 0) + b(\kappa; N + 1, 0, 2) + \dots, \quad (6.3)$$

$$a_{3N+2}(\kappa) = \sum_{l=1}^{2N+2} \sum_{(n_1, n_2) \in J_{3l-2}} b(\kappa; N + l, n_1, n_2) = b(\kappa; N + 1, 0, 1) + \dots, \quad (6.4)$$

where

$$J_l = \{(n_1, n_2) : 2n_1 + n_2 = l\} \subset \mathbb{N}_0 \times \mathbb{N}_0, \quad l = 0, 1, \dots \quad (6.5)$$

Inserting (6.1) and omitting terms of relative order N^{-1} , we get from (6.2)

$$\begin{aligned} a_{3N}(\kappa) &\approx C_{3N}(\kappa) \sum_{l=0}^{2N} \sum_{(n_1, n_2) \in J_{3l}} \sum_{p_1=0}^{n_1} \sum_{p_2=0}^{n_2} e(-\kappa; p_1, p_2) \frac{\Gamma(-\mu(-\kappa) - \frac{2}{3}p_1 - \frac{1}{3}p_2 + l + N)}{\Gamma(-\mu(-\kappa) + N)} \\ &\times (2\kappa s_0)^{-l} \frac{1}{(n_1 - p_1)!} (-2\kappa t_{10})^{n_1 - p_1} \frac{1}{(n_2 - p_2)!} (-2\kappa t_{20})^{n_2 - p_2}, \end{aligned} \quad (6.6)$$

where

$$\begin{aligned} C_{3N}(\kappa) &= -\frac{1}{\pi} \frac{\Gamma(1 + \mu(\kappa) + N) \Gamma(-\mu(-\kappa) + N)}{\Gamma(1 + N)} (2\kappa s_0)^{-N} \\ &\approx -\frac{1}{\pi} \Gamma(\mu(\kappa) - \mu(-\kappa) + N) (2\kappa s_0)^{-N}. \end{aligned} \quad (6.7)$$

Let us now discuss (6.6) in detail: Here only such values of n_1 and n_2 occur for which

$$\frac{2}{3}n_1 + \frac{1}{3}n_2 = l.$$

If again terms of relative order N^{-1} are omitted, the ratio of the gamma functions is equal to

$$N^{-(2/3)p_1 - (1/3)p_2 + l} = N^{(2/3)(n_1 - p_1) + (1/3)(n_2 - p_2)}.$$

The asymptotic behaviour of (6.6) then becomes

$$\begin{aligned} a_{3N}(\kappa) &\approx C_{3N}(\kappa) \sum_{l=0}^{2N} \sum_{(n_1, n_2) \in J_{3l}} \sum_{p_1=0}^{n_1} \sum_{p_2=0}^{n_2} e(-\kappa; p_1, p_2) \\ &\times \frac{1}{(n_1 - p_1)!} [N^{2/3}(2\kappa s_0)^{-2/3}(-2\kappa t_{10})]^{n_1 - p_1} \frac{1}{(n_2 - p_2)!} [N^{1/3}(2\kappa s_0)^{-1/3}(-2\kappa t_{20})]^{n_2 - p_2}. \end{aligned} \quad (6.8)$$

Here the integer power of $2\kappa s_0$ has been splitted in two factors with fractional powers, the meaning of which is given in terms of the phase factor η of (5.10) by

$$(2\kappa s_0)^{-l} = \begin{cases} (2s_0)^{-(2/3)n_1} (2s_0)^{-(1/3)n_2} & \text{if } \kappa = 1 \\ (2s_0)^{-(2/3)n_1} \eta^{2n_1} (2s_0)^{-(1/3)n_2} \eta^{n_2} & \text{if } \kappa = -1 \end{cases}. \quad (6.9)$$

With $n/3$ in place of N , the terms in (6.8) look like the terms of the expansion of the exponential function

$$\begin{aligned} EX_0(n) &:= \exp((\tfrac{1}{3}n)^{2/3}(2\kappa s_0)^{-2/3}(-2\kappa t_{10}) + (\tfrac{1}{3}n)^{1/3}(2\kappa s_0)^{-1/3}(-2\kappa t_{20})) \\ &= 1 + (\tfrac{1}{3}n)^{2/3}(2\kappa s_0)^{-2/3}(-2\kappa t_{10}) + (\tfrac{1}{3}n)^{1/3}(2\kappa s_0)^{-1/3}(-2\kappa t_{20}) + \dots, \end{aligned} \quad (6.10)$$

but because of the three possible values of a third root there are two other such functions,

$$\begin{aligned} EX_1(n) &:= \exp(\eta^4(\tfrac{1}{3}n)^{2/3}(2\kappa s_0)^{-2/3}(-2\kappa t_{10}) + \eta^2(\tfrac{1}{3}n)^{1/3}(2\kappa s_0)^{-1/3}(-2\kappa t_{20})) \\ &= 1 + \eta^4(\tfrac{1}{3}n)^{2/3}(2\kappa s_0)^{-2/3}(-2\kappa t_{10}) + \eta^2(\tfrac{1}{3}n)^{1/3}(2\kappa s_0)^{-1/3}(-2\kappa t_{20}) + \dots, \end{aligned} \quad (6.11)$$

$$\begin{aligned} EX_2(n) &:= \exp(\eta^2(\tfrac{1}{3}n)^{2/3}(2\kappa s_0)^{-2/3}(-2\kappa t_{10}) + \eta^4(\tfrac{1}{3}n)^{1/3}(2\kappa s_0)^{-1/3}(-2\kappa t_{20})) \\ &= 1 + \eta^2(\tfrac{1}{3}n)^{2/3}(2\kappa s_0)^{-2/3}(-2\kappa t_{10}) + \eta^4(\tfrac{1}{3}n)^{1/3}(2\kappa s_0)^{-1/3}(-2\kappa t_{20}) + \dots \end{aligned} \quad (6.12)$$

Therefore (6.8) asymptotically shows exponential behaviour given by a certain linear combination of these three exponential functions. They can be identified by the constant term

and the two linear terms of their expansion shown above. It is convenient, and easy because of the properties (5.11) of η , to introduce three functions which are linear combinations of the exponential functions such that only one of the identifying terms is different from zero,

$$\begin{aligned} L_0(n) &:= \frac{1}{3}[EX_0(n) + EX_1(n) + EX_2(n)] = 1 + \dots, \\ L_1(n) &:= \frac{1}{3}[EX_0(n) + \eta^2 EX_1(n) + \eta^4 EX_2(n)] = (\frac{1}{3}n)^{2/3}(2\kappa s_0)^{-2/3}(-2\kappa t_{10}) + \dots, \\ L_2(n) &:= \frac{1}{3}[EX_0(n) + \eta^4 EX_1(n) + \eta^2 EX_2(n)] = (\frac{1}{3}n)^{1/3}(2\kappa s_0)^{-1/3}(-2\kappa t_{20}) + \dots \end{aligned} \quad (6.13)$$

We can now determine the asymptotic behaviour of (6.8) by looking at the contributions from $(p_1, p_2) = (n_1, n_2)$ or $(p_1, p_2) = (n_1 - 1, n_2)$ or $(p_1, p_2) = (n_1, n_2 - 1)$, respectively. This yields

$$a_{3N}(\kappa) \approx C_{3N}(\kappa) \left\{ S_0(\kappa)L_0(3N) + (2\kappa s_0)^{-1/3}S_1(\kappa)L_1(3N) + (2\kappa s_0)^{-2/3}S_2(\kappa)L_2(3N) \right\}. \quad (6.14)$$

In a similar way we may obtain from (6.3) and (6.4), respectively,

$$\begin{aligned} a_{3N+1}(\kappa) &\approx C_{3N+1}(\kappa) \\ &\times \left\{ (2\kappa s_0)^{-2/3}S_2(\kappa)L_0(3N+1) + S_0(\kappa)L_1(3N+1) + (2\kappa s_0)^{-1/3}S_1(\kappa)L_2(3N+1) \right\}, \end{aligned} \quad (6.15)$$

$$\begin{aligned} a_{3N+2}(\kappa) &\approx C_{3N+2}(\kappa) \\ &\times \left\{ (2\kappa s_0)^{-1/3}S_1(\kappa)L_0(3N+2) + (2\kappa s_0)^{-2/3}S_2(\kappa)L_1(3N+2) + S_0(\kappa)L_2(3N+2) \right\}. \end{aligned} \quad (6.16)$$

If we now switch back to a representation in terms of the exponential functions (6.10)-(6.12), Stokes multipliers, according to (5.13) above, appear as their factors,

$$\begin{aligned} a_{3N}(\kappa) &\approx \frac{1}{3}C_{3N}(\kappa) \\ &\times \{ \sigma_0(\kappa)EX_0(3N) + \sigma_1(\kappa)EX_1(3N) + \sigma_2(\kappa)EX_2(3N) \}, \\ a_{3N+1}(\kappa) &\approx \frac{1}{3}C_{3N+1}(\kappa) \\ &\times \{ \sigma_0(\kappa)EX_0(3N+1) + \eta^2 \sigma_1(\kappa)EX_1(3N+1) + \eta^4 \sigma_2(\kappa)EX_2(3N+1) \}, \\ a_{3N+2}(\kappa) &\approx \frac{1}{3}C_{3N+2}(\kappa) \\ &\times \{ \sigma_0(\kappa)EX_0(3N+2) + \eta^4 \sigma_1(\kappa)EX_1(3N+2) + \eta^2 \sigma_2(\kappa)EX_2(3N+2) \}. \end{aligned} \quad (6.17)$$

These three asymptotic equations can be combined to give

Theorem 4. *The leading terms of the asymptotic exponential behaviour, as $n \rightarrow \infty$, of the coefficients of the formal solutions are given by*

$$\begin{aligned} a_n(\kappa) &\approx -\frac{1}{3}\pi^{-1}(2\kappa s_0)^{-n/3}\Gamma(\frac{1}{3}[\tau(\kappa) - \tau(-\kappa) + n]) \\ &\times [\sigma_0(\kappa)EX_0(n) + \eta^{2n}\sigma_1(\kappa)EX_1(n) + \eta^{4n}\sigma_2(\kappa)EX_2(n)]. \end{aligned} \quad (6.18)$$

This is a concrete example, with all the quantities determined explicitly, of the structural results obtained by Immink [6], and it is interesting also in the context of related work by other authors [5, 15].

7. Postponed proofs

7.1 Proof of Lemma 1

Let

$$f(z) = z^\lambda u(z) \quad (7.1)$$

and multiply the differential equation for $u(z)$ by z^6 in order to remove all the negative powers of z . We then are concerned with the differential equation

$$\begin{aligned} L_z u(z) &:= z^8 u'' + (2\lambda + 1)z^7 u' \\ &- [\sum_{m=1}^6 D_m z^{6-m} + (L^2 - \lambda^2)z^6 + \sum_{m=1}^6 B_m z^{6+m}]u(z) = 0. \end{aligned} \quad (7.2)$$

We are looking for a solution of this differential equation in the form of an integral representation

$$u(z) = \int_{C_{t_2}} \int_{C_{t_1}} \int_{C_s} K(z; s, t_1, t_2) v(s, t_1, t_2) ds dt_1 dt_2 \quad (7.3)$$

with the kernel

$$K = K(z; s, t_1, t_2) = \exp(z^3 s + z^2 t_1 + z t_2). \quad (7.4)$$

If we perform the differentiations with respect to z under the integrals, the differential equation becomes

$$L_z u = \iiint v L_z K ds dt_1 dt_2 = 0, \quad (7.5)$$

where

$$\begin{aligned} \frac{1}{K} L_z K = & (9s^2 - B_6)z^{12} + (12st_1 - B_5)z^{11} + (6st_2 + 4t_1^2 - B_4)z^{10} \\ & + ([6\lambda + 9]s + 4t_1t_2 - B_3)z^9 + ([4\lambda + 4]t_1 + t_2^2 - B_2)z^8 \\ & + ([2\lambda + 1]t_2 - B_1)z^7 + (\lambda^2 - L^2)z^6 - \sum_{m=1}^6 D_m z^{6-m}. \end{aligned} \quad (7.6)$$

It is advisable to rewrite this in terms of the quantities p_1, p_2, p_3 , according to (1.5), which determine the exponential factors of the formal solutions (1.2)-(1.3), or in terms of the related quantities s_0, t_{10}, t_{20} according to (2.3). Then (7.6) becomes

$$\begin{aligned} \frac{1}{K} L_z K = & 9(s^2 - s_0^2)z^{12} + 12(st_1 - s_0t_{10})z^{11} + (6[st_2 - s_0t_{20}] + 4[t_1^2 - t_{10}^2])z^{10} \\ & + ([6\lambda + 9]s + 4[t_1t_2 - t_{10}t_{20}] + [4t_{10}t_{20} - B_3])z^9 \\ & + ([4\lambda + 4]t_1 + [t_2^2 - t_{20}^2] + [t_{20}^2 - B_2])z^8 \\ & + ([2\lambda + 1]t_2 - B_1)z^7 + (\lambda^2 - L^2)z^6 - \sum_{m=1}^6 D_m z^{6-m}. \end{aligned} \quad (7.7)$$

By repeated partial integrations of the exponential function with respect to s or t_1 or t_2 , respectively, it is possible to get rid of all the powers of z . This lengthy task can more conveniently be performed in a formal way as follows: Let us find a partial differential expression $M = M_{s,t_1,t_2}$ with respect to s, t_1, t_2 , independent of z , such that

$$L_z K \equiv M_{s,t_1,t_2} K. \quad (7.8)$$

The powers of z correspond to partial derivatives with respect to s or t_1 or t_2 , and there seems to be some ambiguity as to the choice of the partial derivatives which yield the same total power of z . This ambiguity is resolved by the following reasoning: In order to get the order of each derivative as small as possible, we would prefer derivatives with respect to s , which account for z^3 , with highest priority, next t_1 , which accounts for z^2 , last t_2 , which accounts only for z . Conflicting with this policy, however, are some other requirements which ensure that we get a differential expression appropriate for our purpose. So the singularities should be at the right places, which are related to the coefficients of the exponential factor of the formal solutions we want to represent. In particular, bilinear or quadratic factors, such as t_1^2 for instance, should occur in the form $t_1^2 - t_{10}^2$. For this reason

we have in (7.7) already added and subtracted the term $4t_{10}^2$. It is then necessary to treat these two terms differently, so that any power is always multiplied by the corresponding derivative. In the example just mentioned this means that z^{10} has to be translated into $\partial^4/\partial s^3\partial t_2$ for one term, but into $\partial^4/\partial s^2\partial t_1^2$ for the other. The unique result now is

$$\begin{aligned}
M_{s,t_1,t_2} = & 9(s^2 - s_0^2)\partial^4/\partial s^4 + 12(st_1 - s_0t_{10})\partial^4/\partial s^3\partial t_1 \\
& + 6(st_2 - s_0t_{20})\partial^4/\partial s^3\partial t_2 + 4(t_1^2 - t_{10}^2)\partial^4/\partial s^2\partial t_1^2 \\
& + ([6\lambda + 9]s + [4t_{10}t_{20} - B_3])\partial^3/\partial s^3 + 4[t_1t_2 - t_{10}t_{20}]\partial^4/\partial s^2\partial t_1\partial t_2 \\
& + ([4\lambda + 4]t_1 + [t_{20}^2 - B_2])\partial^3/\partial s^2\partial t_1 + [t_2^2 - t_{20}^2]\partial^4/\partial s^2\partial t_2^2 \\
& + ([2\lambda + 1]t_2 - B_1)\partial^3/\partial s^2\partial t_2 + (\lambda^2 - L^2)\partial^2/\partial s^2 \\
& - D_1\partial^2/\partial s\partial t_1 - D_2\partial^2/\partial s\partial t_2 - D_3\partial/\partial s - D_4\partial/\partial t_1 - D_5\partial/\partial t_2 - D_6.
\end{aligned} \tag{7.9}$$

Next let us introduce the adjoint differential expression $\overline{M} = \overline{M}_{s,t_1,t_2}$ defined, with any sufficiently differentiable function $v = v(s, t_1, t_2)$, by

$$\begin{aligned}
\overline{M}_{s,t_1,t_2}v = & (\partial^4/\partial s^4)\{9(s^2 - s_0^2)v\} + (\partial^4/\partial s^3\partial t_1)\{12(st_1 - s_0t_{10})v\} \\
& + (\partial^4/\partial s^3\partial t_2)\{6(st_2 - s_0t_{20})v\} + (\partial^4/\partial s^2\partial t_1^2)\{4(t_1^2 - t_{10}^2)v\} \\
& - (\partial^3/\partial s^3)\{([6\lambda + 9]s + [4t_{10}t_{20} - B_3])v\} + (\partial^4/\partial s^2\partial t_1\partial t_2)\{4(t_1t_2 - t_{10}t_{20})v\} \\
& - (\partial^3/\partial s^2\partial t_1)\{([4\lambda + 4]t_1 + [t_{20}^2 - B_2])v\} + (\partial^4/\partial s^2\partial t_2^2)\{(t_2^2 - t_{20}^2)v\} \\
& - (\partial^3/\partial s^2\partial t_2)\{([2\lambda + 1]t_2 - B_1)v\} + (\lambda^2 - L^2)(\partial^2v/\partial s^2) \\
& - D_1(\partial^2v/\partial s\partial t_1) - D_2(\partial^2v/\partial s\partial t_2) + D_3(\partial v/\partial s) + D_4(\partial v/\partial t_1) + D_5(\partial v/\partial t_2) - D_6v.
\end{aligned} \tag{7.10}$$

The difference as compared with M is that the factors in front of each derivative have here to be differentiated too and that all the terms of odd order change their sign. The usefulness of the adjoint expression lies in the formula, known in the one-variable case as the identity of Lagrange,

$$vMK - K\overline{M}v = RHS, \tag{7.11}$$

where the right-hand side RHS is a lengthy expression, bilinear in v and K or their partial derivatives, which may be so arranged that it consists of a sum of terms of which each is a total derivative with respect to one of the variables. For the term with the factor $-D_1$,

for instance, this reads

$$v \frac{\partial^2 K}{\partial s \partial t_1} - K \frac{\partial^2 v}{\partial s \partial t_1} = \frac{1}{2} \frac{\partial}{\partial s} \left[v \frac{\partial K}{\partial t_1} - K \frac{\partial v}{\partial t_1} \right] + \frac{1}{2} \frac{\partial}{\partial t_1} \left[v \frac{\partial K}{\partial s} - K \frac{\partial v}{\partial s} \right]. \quad (7.12)$$

By means of (7.5) and (7.8), the equation to be satisfied now is

$$\begin{aligned} L_z u &= \iiint v M_{s,t_1,t_2} K ds dt_1 dt_2 \\ &= \iiint K \overline{M}_{s,t_1,t_2} v ds dt_1 dt_2 + \iiint R H S ds dt_1 dt_2 = 0. \end{aligned} \quad (7.13)$$

The first term in the second line can be made to vanish if v is required to be a solution of the partial differential equation

$$\overline{M}_{s,t_1,t_2} v(s, t_1, t_2) = 0, \quad (7.14)$$

and the second term by a suitable choice of the contours of integration, for it is a sum of semi-integrated terms, each involving the difference of the values of the integrand at the termini of the contour of one variable and only two remaining integrals with respect to the other two variables. The partial differential equation (2.2) is the same as (7.14), after the derivatives of the products in (7.10) have been resolved. This completes the proof of Lemma 1.

7.2 Proof of Lemma 2

In terms of the shifted variables

$$S = s - \kappa s_0, \quad T_1 = t_1 - \kappa t_{10}, \quad T_2 = t_2 - \kappa t_{20}, \quad (7.15)$$

the differential equation (2.2) reads

$$\begin{aligned} &9S(S + 2\kappa s_0)(\partial^4 v / \partial S^4) + 12(ST_1 + \kappa t_{10}S + \kappa s_0 T_1)(\partial^4 v / \partial S^3 \partial T_1) \\ &+ 6(ST_2 + \kappa t_{20}S + \kappa s_0 T_2)(\partial^4 v / \partial S^3 \partial T_2) + 4T_1(T_1 + 2\kappa t_{10})(\partial^4 v / \partial S^2 \partial T_1^2) \\ &+ T_2(T_2 + 2\kappa t_{20})(\partial^4 v / \partial S^2 \partial T_2^2) + 4(T_1 T_2 + \kappa t_{20} T_1 + \kappa t_{10} T_2)(\partial^4 v / \partial S^2 \partial T_1 \partial T_2) \\ &+ ([81 - 6\lambda]\kappa s_0 + B_3 - 4t_{10}t_{20})(\partial^3 v / \partial S^3) + (81 - 6\lambda)S(\partial^3 v / \partial S^3) \\ &+ ([52 - 4\lambda]\kappa t_{10} + B_2 - t_{20}^2)(\partial^3 v / \partial S^2 \partial T_1) + (52 - 4\lambda)T_1(\partial^3 v / \partial S^2 \partial T_1) \\ &+ ([25 - 2\lambda]\kappa t_{20} + B_1)(\partial^3 v / \partial S^2 \partial T_2) + (25 - 2\lambda)T_2(\partial^3 v / \partial S^2 \partial T_2) \\ &+ ([\lambda - 12]^2 - L^2)(\partial^2 v / \partial S^2) - D_1(\partial^2 v / \partial S \partial T_1) - D_2(\partial^2 v / \partial S \partial T_2) \\ &+ D_3(\partial v / \partial S) + D_4(\partial v / \partial T_1) + D_5(\partial v / \partial T_2) - D_6 v = 0. \end{aligned} \quad (7.16)$$

Inserting a power series solution

$$v = \sum_{m=0} \sum_{n_1=0} \sum_{n_2=0} a(m, n_1, n_2) S^{\mu+m} T_1^{-\nu_1-n_1} T_2^{-\nu_2-n_2}, \quad (7.17)$$

we obtain for the coefficients the recurrence relation

$$\begin{aligned} & 6\kappa s_0(\mu+m)(\mu+m-1)(\mu+m-2)[3(\mu+m)-2(\nu_1+n_1)-(\nu_2+n_2)-\lambda-\tau(\kappa)+6] \\ & \times a(m, n_1, n_2) \\ & + (\mu+m-1)(\mu+m-2)\{[3(\mu+m)-2(\nu_1+n_1)-(\nu_2+n_2)-\lambda+3]^2-L^2\} \\ & \times a(m-1, n_1, n_2) \\ & + (\mu+m-1)(\mu+m-2)(\nu_1+n_1-1)\{4\kappa t_{10}[-3(\mu+m)+2(\nu_1+n_1)+(\nu_2+n_2) \\ & + \lambda-4] + t_{20}^2 - B_2\}a(m-1, n_1-1, n_2) \\ & + (\mu+m-1)(\mu+m-2)(\nu_2+n_2-1)\{2\kappa t_{20}[-3(\mu+m)+2(\nu_1+n_1)+(\nu_2+n_2) \\ & + \lambda-\frac{7}{2}] + B_1\}a(m-1, n_1, n_2-1) \\ & + D_1(\mu+m-2)(\nu_1+n_1-1)a(m-2, n_1-1, n_2) \\ & + D_2(\mu+m-2)(\nu_2+n_2-1)a(m-2, n_1, n_2-1) \\ & + D_3(\mu+m-2)a(m-2, n_1, n_2) - D_4(\nu_1+n_1-1)a(m-3, n_1-1, n_2) \\ & - D_5(\nu_2+n_2-1)a(m-3, n_1, n_2-1) - D_6a(m-3, n_1, n_2) = 0, \end{aligned} \quad (7.18)$$

valid for $m \geq 0$, $n_1 \geq 0$, $n_2 \geq 0$ provided that we agree that all the a -coefficients are equal to zero if any of the indices m, n_1, n_2 is less than zero. Assuming that $a(0, 0, 0) \neq 0$, we get from the equation for $m = n_1 = n_2 = 0$ the indicial equation

$$\mu(\mu-1)(\mu-2)(3\mu-2\nu_1-\nu_2-\lambda-\tau(\kappa)+6) = 0, \quad (7.19)$$

with $\tau(\kappa)$ according to (1.6). Possible values of the exponent μ are therefore $\mu = 0, 1, 2$, or

$$\mu = \frac{1}{3}[2\nu_1 + \nu_2 + \lambda + \tau(\kappa)] - 2. \quad (7.20)$$

In order to avoid complications, we may assume that the last possibility does not yield an integer value. This can always be guaranteed by a suitable choice of the still disposable parameter λ . Inspection of the recurrence relation then shows that each of the possible

values of μ leads to a solution of the partial differential equation. We here need not further consider the three solutions which are regular with respect to S at $S = 0$, but we continue to discuss the singular one with the exponent (7.20), writing for the associated coefficients $A(\kappa; m, n_1, n_2)$ rather than $a(m, n_1, n_2)$. Simplifying by means of (7.20) and introducing the b -coefficients according to (2.6), we get (2.7). The coefficients for which $3m - 2n_1 - n_2 = 0$ are constants of integration and may be chosen to be zero. This completes the proof of Lemma 2.

7.3 Proof of the limit formula

We have to verify that the limit formula (4.14) satisfies (4.13). Substituting it for the e -coefficients in (4.13) and interchanging the summations, we have to evaluate sums such as

$$\sum_{p=j}^n \frac{(-1)^{n-p}}{(p-j)!(n-p)!} = \frac{(-1)^{n-j}}{(n-j)!} \sum_{q=0}^{n-j} (-1)^q \binom{n-j}{q} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{if } 0 \leq j \leq n-1, n > 0 \end{cases} \quad (7.21)$$

Therefore only one term survives on the right-hand side, which then becomes equal to the left.

7.4 Proof of Theorem 1

In order to prove Theorem 1, we first multiply the continuation formula (3.5), with $r_1 = r_2 = 0$, by

$$\left(\frac{1}{2} - \frac{s}{2\kappa s_0}\right)^{-\mu(\kappa)} = \sum_{j=0}^{\infty} \frac{(\mu(\kappa))_j}{j!} \left(\frac{1}{2} + \frac{s}{2\kappa s_0}\right)^j, \quad (7.22)$$

where the left-hand side is used on the left and the right-hand side on the right of the continuation formula. Then the left-hand side is the same as above (4.1) with the power factor in front of the series removed, but the right becomes, after multiplication of the two power series,

$$\begin{aligned} & \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} E(-\kappa; 0, 0; q_1, q_2) \Gamma(-\mu(-\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2) \\ & \times \sum_{k=0}^{\infty} \sum_{l_1=0}^k \sum_{l_2=0}^k \Gamma(\nu_1 + q_1 + l_1) \Gamma(\nu_2 + q_2 + l_2) H(-\kappa; k, l_1, l_2; q_1, q_2) \\ & \times \left(\frac{1}{2} + \frac{s}{2\kappa s_0}\right)^{\mu(-\kappa) + \frac{2}{3}q_1 + \frac{1}{3}q_2 + k} (t_1 + \kappa t_{10})^{-\nu_1 - q_1 - l_1} (t_2 + \kappa t_{20})^{-\nu_2 - q_2 - l_2}, \end{aligned} \quad (7.23)$$

with H defined in (4.16). Proceeding as above, and making use of the formula

$$(\alpha - k)_m = (\alpha)_m \frac{(1 - \alpha)_k}{(1 - \alpha - m)_k}, \quad (7.24)$$

we obtain

$$\begin{aligned} & \frac{-\pi}{\sin(\pi\mu(\kappa))} \frac{1}{\Gamma(1 + \mu(\kappa) + m)} (2\kappa s_0)^m b(\kappa; m, n_1, n_2) \\ & \sim \sum_{q_1=0}^{n_1} \sum_{q_2=0}^{n_2} E(-\kappa; 0, 0; q_1, q_2) \frac{\Gamma(-\mu(-\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2 + m)}{m!} \\ & \times \sum_{k=0}^{\infty} \sum_{l_1=0}^k \sum_{l_2=0}^k \frac{(1 + \mu(-\kappa) + \frac{2}{3}q_1 + \frac{1}{3}q_2)_k}{(1 + \mu(-\kappa) + \frac{2}{3}q_1 + \frac{1}{3}q_2 - m)_k} H(-\kappa; k, l_1, l_2; q_1, q_2) \\ & \times \left(\frac{1}{j_1!} (-2\kappa t_{10})^{j_1} \right)_{j_1+q_1+l_1=n_1} \left(\frac{1}{j_2!} (-2\kappa t_{20})^{j_2} \right)_{j_2+q_2+l_2=n_2}. \end{aligned} \quad (7.25)$$

Keeping the first $K + 1$ singular terms on the right and solving for the E -coefficient with $q_1 = n_1, q_2 = n_2$, we have

$$\begin{aligned} & E(-\kappa; 0, 0; n_1, n_2) \\ & \times \left[1 + \sum_{k=1}^K \frac{(1 + \mu(-\kappa) + \frac{2}{3}n_1 + \frac{1}{3}n_2)_k}{(1 + \mu(-\kappa) + \frac{2}{3}n_1 + \frac{1}{3}n_2 - m)_k} H(-\kappa; k, 0, 0; n_1, n_2) + O(m^{-K-1}) \right] \\ & = \frac{-\pi}{\sin(\pi\mu(\kappa))} \frac{m!}{\Gamma(1 + \mu(\kappa) + m) \Gamma(-\mu(-\kappa) - \frac{2}{3}n_1 - \frac{1}{3}n_2 + m)} (2\kappa s_0)^m b(\kappa; m, n_1, n_2) \\ & - \sum_{\substack{q_1=0 \\ (q_1, q_2) \neq (n_1, n_2)}}^{n_1} \sum_{q_2=0}^{n_2} E(-\kappa; 0, 0; q_1, q_2) \frac{\Gamma(-\mu(-\kappa) - \frac{2}{3}q_1 - \frac{1}{3}q_2 + m)}{\Gamma(-\mu(-\kappa) - \frac{2}{3}n_1 - \frac{1}{3}n_2 + m)} \\ & \times \left[1 + \sum_{k=1}^K \sum_{l_1=0}^k \sum_{l_2=0}^k \frac{(1 + \mu(-\kappa) + \frac{2}{3}q_1 + \frac{1}{3}q_2)_k}{(1 + \mu(-\kappa) + \frac{2}{3}q_1 + \frac{1}{3}q_2 - m)_k} H(-\kappa; k, l_1, l_2; q_1, q_2) + O(m^{-K-1}) \right] \\ & \times \left(\frac{1}{j_1!} (-2\kappa t_{10})^{j_1} \right)_{j_1+q_1+l_1=n_1} \left(\frac{1}{j_2!} (-2\kappa t_{20})^{j_2} \right)_{j_2+q_2+l_2=n_2}. \end{aligned} \quad (7.26)$$

This is essentially (4.15) because of (4.11) and completes the proof of Theorem 1.

7.5 Choice of the computational parameter λ

If $D_3 = D_6 = 0$, the recurrence relation (2.7) for $b(\kappa; m, 0, 0)$ reduces to a two-term relation, and we obtain

$$b(\kappa; m, 0, 0) = (2\kappa s_0)^{-m} \frac{(-\frac{1}{3}L + \frac{1}{3}\tau(\kappa))_m (\frac{1}{3}L + \frac{1}{3}\tau(\kappa))_m}{m!}. \quad (7.27)$$

Rewriting (4.16) by means of the identity

$$\frac{(x)_{k-j}}{(k-j)!} = \frac{(x)_k}{k!} \frac{(-k)_j}{(1-x-k)_j}, \quad (7.28)$$

we then have

$$H(-\kappa; k, 0, 0; 0, 0) = \frac{(\mu(\kappa))_k}{k!} \sum_{j=0}^k \frac{(-k)_j (-\frac{1}{3}L + \frac{1}{3}\tau(-\kappa))_j (\frac{1}{3}L + \frac{1}{3}\tau(-\kappa))_j}{(1 - \mu(\kappa) - k)_j (1 + \mu(-\kappa))_j j!}. \quad (7.29)$$

Because of $3\mu(\kappa) = \lambda + \tau(\kappa) - 3$ according to (2.5) and (2.11) and $\tau(\kappa) = 3 - \tau(-\kappa)$ according to (1.6), the series is a terminating one-balanced hypergeometric series at unit argument, which can be summed by the theorem of Saalschütz [11], so that

$$H(-\kappa; k, 0, 0; 0, 0) = \frac{(-\frac{1}{3}L + \frac{1}{3}\lambda)_k (\frac{1}{3}L + \frac{1}{3}\lambda)_k}{(1 + \mu(-\kappa))_k k!}. \quad (7.30)$$

In total, we get

$$\begin{aligned} e(-\kappa; 0, 0) &= \frac{-\pi}{\Gamma(-\frac{1}{3}L + \frac{1}{3}\tau(\kappa))\Gamma(\frac{1}{3}L + \frac{1}{3}\tau(\kappa))} \frac{\Gamma(-\frac{1}{3}L + \frac{1}{3}\tau(\kappa) + m)\Gamma(\frac{1}{3}L + \frac{1}{3}\tau(\kappa) + m)}{\Gamma(\frac{1}{3}\lambda + \frac{1}{3}\tau(\kappa) + m)\Gamma(-\frac{1}{3}\lambda + \frac{1}{3}\tau(\kappa) + m)} \\ &\times \left[1 + \sum_{k=1}^K \frac{(-\frac{1}{3}L + \frac{1}{3}\lambda)_k (\frac{1}{3}L + \frac{1}{3}\lambda)_k}{(\frac{1}{3}\lambda + \frac{1}{3}\tau(-\kappa) - m)_k k!} + O(m^{-K-1}) \right]^{-1}. \end{aligned} \quad (7.31)$$

For $\lambda = -L$ or $\lambda = L$, the terms with $k = 1, 2, \dots$ of the asymptotic series all vanish and the factor in front of the series becomes independent of m . More generally this means that some terms which might become quite large when L is not small can be removed by such a choice of λ from the asymptotic series and incorporated in the m -dependence of the function in front of the series.

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